A list of statements/theorems that you should be able to prove, together with the main idea of the proof for some of them.

1. If $f : X \to Y$ is a map and $A_1, A_2 \subset X$, $B_1, B_2 \subset Y$ are subsets, then
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2).$$

2. If A and B are countable sets, then $A \times B$ is countable.
[If you write the elements of $A \times B$ in a grid, you can list them by following diagonals.]

3. The set of rationals is countable.
[You can write $\mathbf{Q}$ as a subset of $\mathbf{Z} \times \mathbf{Z}$ and use the previous result]

4. The set of infinite sequences whose elements are all 0 or 1 is uncountable.
[If you list a countable collection of such sequences, you can always construct a new sequence which differs from all the ones in the list]

5. Suppose that M, N are subsets of a metric space X. The closure operation satisfies the following properties:
   (a) If $M \subset N$ then $[M] \subset [N]$.
   (b) $[[M]] = [M]$.
   (c) $[M \cup N] = [M] \cup [N]$.
   (d) $[\emptyset] = \emptyset$.

6. Closed and open subsets of a metric space satisfy the following properties:
   (a) The intersection of an arbitrary collection of closed sets is closed.
   (b) The union of finitely many closed sets is closed.
   (c) The intersection of finitely many open sets is open.
   (d) The union of an arbitrary collection of open sets is open.

7. A subset $M \subset R$ in a metric space R is open if and only if the complement $R \setminus M$ is closed.
[A point $x \in M$ is an interior point of M if and only if x is not a contact point for $R \setminus M$.]

8. Every convergent sequence in a metric space is a Cauchy sequence.
[If $\rho(x_n, x) < \epsilon/2$ and $\rho(x_m, x) < \epsilon/2$ then $\rho(x_n, x_m) < \epsilon$ by the triangle inequality.]

9. (Nested sphere theorem) Let R be a complete metric space, and $\overline{B}_{r_k}(x_k)$ a sequence of closed balls in R such that
$$\overline{B}_{r_1}(x_1) \supset \overline{B}_{r_2}(x_2) \supset \dots,$$
and $r_k \to 0$ as $k \to \infty$. Then the intersection $\bigcap_{k \geq 1} \overline{B}_{r_k}(x_k)$ is non-empty.
[Show that $\{x_k\}$ is a Cauchy sequence. The limit must be in all of the balls because they are closed.]

10. (Baire's theorem) Let R be a complete metric space, and suppose that

$$R = \bigcup_{k=1}^{\infty} A_k$$

for a sequence of subsets $A_k \subset R$. Then it is not possible that all of the A_k are nowhere dense.
[It a set A is nowhere dense, then every closed ball $\overline{B}$ in R contains a smaller closed ball $\overline{B}' \subset \overline{B}$ which is disjoint from A. Use this to obtain a nested sequence of closed balls $\overline{B}_1 \supset \overline{B}_2 \supset \ldots$, such that $\overline{B}_k$ is disjoint from A_k, and apply the nested sphere theorem.]

11. Let R be a complete metric space, and $F : R \to R$ a map such that for some $\alpha < 1$ we have

$$\rho(F(x), F(y)) \leq \alpha\rho(x, y)$$

for all $x, y \in R$. Then there exists a unique point $x \in R$ for which $F(x) = x$.
[Pick any $x_0 \in R$ and define x_n inductively by $x_n = F(x_{n-1})$. Use the contraction property to show that this sequence is a Cauchy sequence, and the limit is a fixed point of F.]

12. A map $f : X \to Y$ between topological spaces is continuous, if and only if $f^{-1}(U)$ is open for every open set $U \subset Y$.

13. The interval $[0, 1]$ is connected.
[If $U \subset [0, 1]$ is non-empty, open and closed, then show that $\sup U = 1$, and so $1 \in U$. The same is true for the complement of U. So we cannot write $[0, 1]$ as the disjoint union of two non-empty open subsets.]

14. If a topological space X is path connected, then it is connected.
[If X were not connected, then $X = U \cup V$ for two non-empty disjoint open subsets. Get a contradiction by connecting points $x \in U$ and $y \in V$ using a path, and using that $[0, 1]$ is connected.]

15. Let $f : X \to Y$ be continuous, and suppose that X is connected. Then $f(X)$ is connected.
[If $f(X) = U \cup V$ with U and V disjoint, then $X = f^{-1}(U) \cup f^{-1}(V)$ with $f^{-1}(U)$ and $f^{-1}(V)$ disjoint.]

16. A topological space X is compact if and only if every centered system of closed sets in X has non-empty intersection.

17. If X is compact and $F \subset X$ is closed, then F is compact.
[If $\{U_\alpha\}$ is an open cover of F, then $\{U_\alpha\}$ together with $X \setminus F$ gives an open cover of X]

18. Suppose that X is a Hausdorff space, and $K \subset X$ is compact. Then K is closed.
[Suppose $y \notin K$. For all $x \in K$ there are disjoint open neighborhoods $x \in U_x$ and $y \in V_x$. Finitely many of the U_x cover K, so the intersection of the corresponding sets V_x is a neighborhood of y disjoint from K.]

19. Let X be compact, and $f : X \to Y$ continuous. Then $f(X)$ is compact.
[If $\{U_\alpha\}$ is an open cover of $f(X)$, then $\{f^{-1}(U_\alpha)\}$ is an open cover of X.]

20. Any sequence in a compact metric space has a convergent subsequence.
[For a sequence $\{x_n\}$ consider the sets $E_k = \{x_k, x_{k+1}, \ldots\}$. Then the closures $[E_k]$ form a centered system.]

21. A metric space R is compact if and only if it is complete and totally bounded.
[If $\{U_\alpha\}$ give an open cover of R which has no finite subcover, then the total boundedness can be used to obtain a nested sequence $A_1 \supset A_2 \supset \ldots$ of closed subsets of R with diameters going to zero, and such that no A_i is covered by a finite subcollection of the $\{U_\alpha\}$. Get a contradiction by looking at a point in the intersection of all the A_i.]

22. A subset $M \subset C_{[0,1]}$, with the distance $\rho(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$ is totally bounded, if and only if it is uniformly bounded and equicontinuous.
[If M is totally bounded, then for any $\epsilon > 0$ you can find functions $f_1, \ldots, f_k$ approximating any element of M within ϵ. The uniform continuity of the f_i can then be used to prove equicontinuity of M.
To show total boundedness of M, use piecewise linear approximations to approximate functions in M.]

23. If $F : X \to \mathbf{R}$ is continuous, and the metric space X is compact, then F is uniformly continuous.
[Fix $\epsilon > 0$, and for every x find a ball B_{r_x} in which F can change at most ϵ (using continuity). Cover X by finitely many of the balls $B_{r_x/2}$ with half the radius, and show that if $\rho(y,z) < r/2$, where r is the smallest radius, then $y, z \in B_{r_x}$ for some x.]

24. If $F : X \to \mathbf{R}$ is continuous and X is compact, then $F(X)$ is bounded, and F achieves its infimum and supremum.
[$F(X)$ is compact, so it is closed and bounded.]

25. If $F : X \to \mathbf{R}$ is lower semicontinuous and X is compact, then $F(X)$ is bounded from below, and F achieves its infimum.
[For every x, we have $f(y) > f(x) - 1$ for y in a neighborhood of x. Find a finite cover with such neighborhoods to get a lower bound for $F(X)$. To show that the infimum is achieved, find a sequence x_n such that $F(x_n) \to \inf F(X)$, and show that a subsequence of $\{x_n\}$ converges to some x with $F(x) = \inf F(X)$.]

26. If V is a normed linear space and $W \subset V$ is a closed subspace, then the quotient V/W is also a normed linear space with the norm

$$\|[x]\|_{V/W} = \inf\{\|x - y\|_V \,;\, y \in W\}.$$

[The main thing to check is that if $\|[x]\| = 0$, then $x \in W$. This uses that W is closed.]

27. If a linear functional $f : V \to \mathbf{R}$ on a normed linear space is continuous at a point, then it is continuous everywhere.
[If f is continuous at a point x_0, and you want to check continuity at a point x, then use that $f(y) - f(x) = f(x_0 + y - x) - f(x_0)$, since f is linear. So if $\|y - x\|$ is small, you can use the continuity at x_0 to control $f(y) - f(x)$.]

28. A linear functional $f : V \to \mathbf{R}$ on a normed linear space is continuous if and only if it is bounded.

    [Bounded $\Rightarrow$ Continuous is easy. Conversely by continuity at 0, there is a $\delta > 0$ such that $\|f(x)\| < 1$ whenever $\|x\| < \delta$. Using the linearity of f this shows that f is bounded.]

29. If V is a normed linear space, then its conjugate space V^* is complete, i.e. it is a Banach space.

    [Similar to showing that $C_{[a,b]}$ is complete with respect to the sup norm. If f_n is a Cauchy sequence in V^*, then show that $f_n(x)$ is a Cauchy sequence of real numbers for every $x \in V$. This way you can define a limit $f(x) = \lim_{n\to\infty} f_n(x)$. One still needs to show that $f \in V^*$, and that $f_n \to f$ in V^*.]

30. If V is a normed linear space and $x \in V$ a non-zero element, then there is a continuous linear functional $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

    [You can first find such an f on the span of x, and then extend it to all of V using the Hahn-Banach theorem.]